Int. J. Solids Structures Vol. 22, No. 2, pp. 209–223, 1986
Printed in Great Britain.

0020–7683/86 $3.00 + .00

OPTIMAL LAYOUT OF LONG-SPAN TRUSS-GRIDS—I†

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(*Received* 17 *June* 1984; *in revised form* 4 *March* 1985)

Abstract—After reviewing briefly Prager's optimal-layout theory, the optimization of isolated trusses is considered, and then optimality criteria are derived for the minimum-weight layout of truss-grids. The theory is applied to simply supported circular truss-grids, whose radius does not exceed a certain limiting value. Longer spans and other boundary conditions will be considered in Part II of this study, in which weight comparisons for various solutions are also given. It will be shown that layout optimization for long-span systems results in most substantial savings. In all problems considered, the optimal solution is derived *analytically* and confirmed by three independent methods: namely, primal (variational) formulation, dual formulation and direct minimization of the cost with respect to some geometrical parameters of the layout. Although the problems discussed in this paper are formulated in the context of optimal *plastic* design, the solutions obtained are statically determinate and hence they are also valid in optimal *elastic* design for given permissible stresses.

INTRODUCTION

Optimization of the layout of long-span surface-structures is of practical significance for two reasons: first, the self-weight of these systems constitutes a major part of the load; second, the variation of the total weight is highly sensitive to changes in the layout. Numerical methods are not very efficient in optimizing the layout of complex grid-like systems because (a) a very large number of feasible members would have to be considered for a reasonable accuracy, and (b) the number of local minima increases exponentially with the number of potential members.

By using a continuum-type approximation and Prager's layout theory, however, the problem is rendered convex, and hence the solution can be obtained by a direct and systematic procedure.

This paper is concerned with the optimal layout of grillages consisting of a system of intersecting vertical trusses whose depth is prescribed. Unlike earlier studies[1, 2], the effect of both bending moment and shear force on the specific truss-weight is taken into consideration.

Although the underlying theory was originally based on optimal plastic design, least-weight plastic solutions for a single external load system (and even for additional self-weight) often turn out to be kinematically admissible, and hence they are also valid in optimal elastic design. In fact, Rozvany[3] and Olhoff[4] have shown that the optimal layout of plastically designed grillages also minimizes the total weight for given permissible stress[3], given compliance[3] or given natural frequency[4] in elastic design.

The optimality condition used in this paper was obtained in its original form by Prager and Shield[5] and extended to self-weight by Rozvany[6]. An equivalent of the Prager–Shield criterion can also be derived from theories of Masur[7], Mroz[8] or Save[9].

In all solutions to be discussed, the minimum structural weight is calculated independently from both primal and dual formulation. This procedure provides a reliable check on optimality.

The current investigation forms part of a broader project on optimal long-span surface-structures which was initiated by the late Professor W. Prager†. Other long-

† This project was initiated by the late Professor W. Prager (Brown University). The first stage of this project was carried out by Prof. Prager and G. I. N. Rozvany under a government-sponsored research contract (SFB-64) in Stuttgart, West Germany.

span structures investigated recently by Rozvany's research group include archgrids and cable networks[10–14] as well as shell roofs[15–17]. Another important investigation concerning the effect of self-weight on least-weight *elastic* structures was carried out by Karihaloo and Hemp[18], and opens up new avenues in this important area of research. One of the latest developments in optimal-layout theory concerns constraints on geometrical gradients[19]. This idea was pioneered recently, in the context of elastic solid plates, by Niordson[20].

The current paper demonstrates that relatively sophisticated mathematical methods can furnish results of practical significance in branches of structural mechanics in which numerical methods have not been very successful. In Part I, Prager's optimal-layout theory is reviewed, optimization of a single truss discussed and optimality criteria for long-span truss-grids presented. As an example, the optimal layout of circular simply supported trusses within a limiting radius is derived. Part II discusses the optimal layout of circular trusses having (i) simple supports and very long spans, or (ii) built-in edges. Finally a weight comparison for various solutions is given.

A BRIEF REVIEW OF PRAGER'S LAYOUT THEORY

As Prager pointed out[21], optimization of the structural layout is a particularly "challenging" problem because it involves an infinite number of possible topographies or configurations. In solving such problems, Prager introduced a rather ingenious idea which converts this complex problem of *optimization* into that of elastic (nonlinear) *analysis*. This approach was then generalized and extended considerably by Rozvany's research group.

Prager's layout theory is based on two underlying concepts, viz. the Prager–Shield theory of optimal plastic design[3, 5] and the notion of "structural universe"[22–24].

In optimal plastic design[5], the "specific cost" ψ, i.e. cost (or weight) per unit length, area or volume can be expressed in terms of the "generalized" stresses (local stresses or stress resultants) **Q**, and then the total "cost" Φ is to be minimized subject to statical admissibility (^{s}):

$$\min_{\mathbf{Q}^s} \Phi = \int_D \psi(\mathbf{Q})\, \mathrm{d}\mathbf{x}, \tag{1}$$

where D is the structural domain referred to coordinates **x**.

With a view to illustrating the above concepts, consider a rectangular beam of given depth d, but continuously varying width b. Then the plastic moment capacity is $\pm M_p = \sigma_y b d^2/4$, where σ_y is the yield stress. Since the beam weight per unit length is $\psi = \gamma b d$ where γ is the specific weight of the beam material, the "specific cost function" can be expressed as $\psi = k\,|\,M\,|$ with $k = 4\gamma/\sigma_y d$. In this problem, the "generalized" stress Q is the bending moment M. The total "cost" (weight) then becomes $\Phi = \int_0^L k\,|\,M\,|\,\mathrm{d}x$, where x is the distance along the beam axis and L is the beam length. In designing a beam plastically, only *static admissibility* needs to be satisfied. The latter consists of the equilibrium condition [$\mathrm{d}^2M/\mathrm{d}x^2 = -p(x)$ where $p(x)$ is the vertical load] and static boundary conditions (e.g. $M = 0$ at free ends and simple supports).

The Prager–Shield condition[3, 5] converts the problem in eqn (1) into a strain–stress relationship

$$\text{on } D, \qquad \max_{\mathbf{q}^k} = G[\psi(\mathbf{Q}^s)], \tag{2}$$

where $\mathbf{q}^k$ is the "generalized" strain vector, (^{k}) denotes kinematic admissibility, and **G** is the subgradient[3, 25]. The condition (2) is a necessary and sufficient one for convex specific cost function $\psi(Q)$ with linear equilibrium equations, and converts a problem of optimal design into a problem of elastic analysis.

The strain field **q** and corresponding displacement field **u** furnished by eqn (2) are

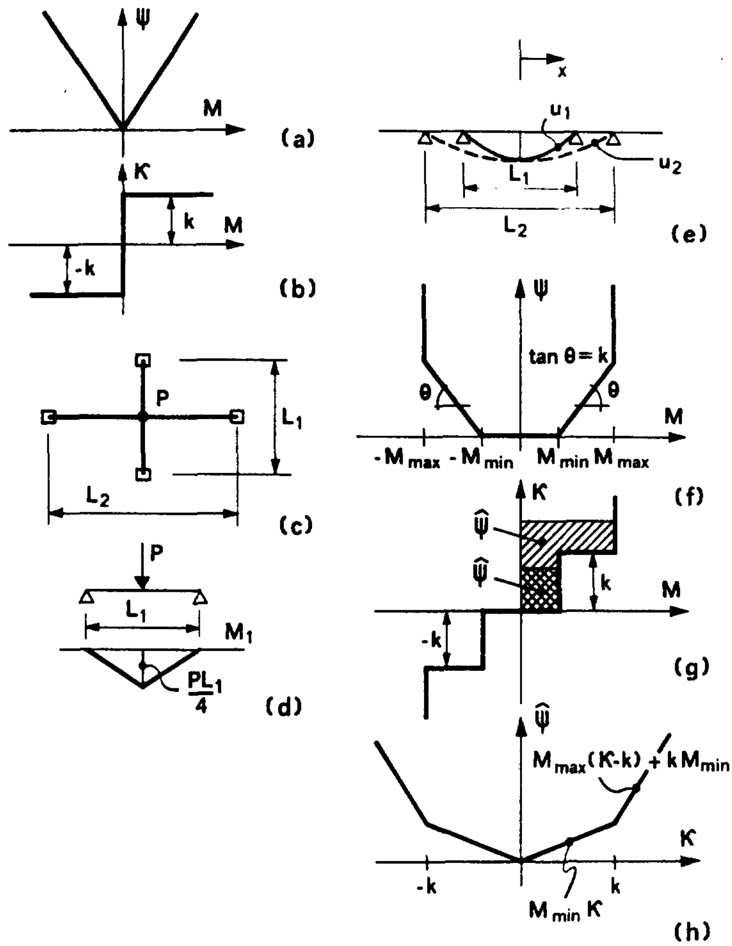

Fig. 1. Review of optimal-layout theory.

fictitious quantities which facilitate greatly the optimization procedure but do not necessarily represent the elastic displacements of the system. To distinguish them from the latter, displacements arising from static-kinematic optimality criteria shall be termed *Pragerian displacement fields*.

Considering the above beam example again, the relevant generalized strain q is the beam curvature $\kappa = -d^2u/dx^2$ where u is the Pragerian beam deflection. The subgradient G denotes a collection of first derivatives with respect to the stress components $G[\psi(\mathbf{Q})] = (\partial\psi/\partial Q_1, \partial\psi/\partial Q_2, \ldots, \partial\psi/\partial Q_n)$ if $\psi(\mathbf{Q})$ is differentiable at the stress $\mathbf{Q}$ considered. At slope discontinuities, however, any *convex combination* of the slopes for the adjacent "regimes" may be taken[3, 25]. (A stress regime is a set of stress values, on the interior of which $\psi(\mathbf{Q})$ is differentiable.) Considering the specific cost function $\psi = k \mid M \mid$ in the beam example, the specific cost function is shown graphically in Fig. 1(a) and the corresponding subgradient $\kappa = G[\psi(M)]$ in Fig. 1(b). The same relation can be expressed analytically as $\kappa = k$ (for $M > 0$), $\kappa = -k$ (for $M < 0$), $-k \le \kappa \le k$ (for $M = 0$). Note that $\kappa = G[\psi(M)]$ is nonunique at $M = 0$.

The *structural universe* consists of all feasible members[22–24]. Since the Prager–Shield condition [see eqn (2)] also gives a (usually nonunique) strain requirement for a zero generalized stress vector [i.e. vanishing or nonoptimal members, see $M = 0$ in Fig. 1(b)], its fulfillment for the entire structural universe constitutes a necessary and sufficient condition of layout optimality for convex specific cost functions. The same problem becomes usually *nonconvex* if it is expressed in terms of unknown geometrical parameters. An advantage of Prager's layout theory is therefore the preservation of convexity which is achieved by embedding the problem into a structural universe. Moreover, this method automatically eliminates nonoptimal members from the solution. Figure 1(c), for example, shows in plan view a very simple structural universe consisting of two potential (or "candidate") members (intersecting simply supported beams with a central point load). Using the specific cost function in Fig. 1(a), the optimal solution is clearly the case when the short beam (with the span L_1) carries the entire load and is subjected to positive moments throughout [Fig. 1(d)]. Then the optimal moment-curvature relation in Fig. 1(b) gives $\kappa = k$, $u_1 = k(L_1^2/8 - x^2/2)$ for the short beam [Fig. 1(e)]. The curvature for the longer beam [broken line in Fig. 1(e)] clearly satisfies the condition $\kappa \leq k$ for $M \equiv 0$ [Fig. 1(b)] with $u_2 = k(L_1^2/L_2^2)(L_2^2/8 - x^2/2)$, and thus the optimality of the above solution is established. Naturally, the optimal solution in the above example is intuitively obvious.

It was shown by Rozvany[6] that an extended version of the Prager–Shield condition automatically takes *self-weight* (dead load) into consideration if we modify the original condition in the following form

$$\text{on } D, \qquad \mathbf{q}^k = (1 + u^k)\mathbf{G}[\psi(\mathbf{Q}^s)], \tag{3}$$

where u^k is the vertical Pragerian displacement.

In convex problems, the minimum total cost $\Phi_{\min}$ can be derived from *primal formulation* (1) or from *dual formulation*

$$\Phi_{\min} = \max_{\mathbf{u}^k} \Phi^* = \int_D [\mathbf{up} - \hat{\psi}(\mathbf{q})]\, d\mathbf{x}, \tag{4}$$

where $\mathbf{u}^k(\mathbf{x})$ is a kinematically admissible displacement, $\mathbf{p}(\mathbf{x})$ is the load and $\hat{\psi}$ is the "complementary cost." Considering any strain field $\mathbf{q}$ satisfying the relation under eqn (2), the complementary cost is given by

$$\hat{\psi} = \int_0^q \mathbf{Q}\cdot d\mathbf{q}.$$

To illustrate the notion of complementary cost, consider again a beam having a cost function $k\,|\,M\,|$, but with prescribed minimum and maximum cross-sections $[kM_{\min}, kM_{\max}]$. If the modified specific cost ψ is the cross-sectional area in excess of $kM_{\min}$, then the corresponding functional relation $\psi(M)$ is the one shown in Fig. 1(f) and the strain–stress relation furnished by the Prager–Shield condition (2) is shown in Fig. 1(g). It can be seen from the latter that in this problem

$$\hat{\psi} = \int_0^\kappa M d\kappa = \kappa M_{\min} \qquad (\text{for } \kappa \leq k)$$

and

$$\hat{\psi} = kM_{\min} + (\kappa - k)M_{\max} \qquad (\text{for } \kappa \geq k)$$

[(Fig. 1(h)].

Before considering the layout optimization of truss-grids, the application of the modified Prager–Shield condition (3) will be illustrated with an example involving a single beam (truss).

AN INTRODUCTORY EXAMPLE

Consider a built-in beam [Fig. 2(a)] subject to a central-point load $2\bar{P}$ and its own weight. It is assumed that the self-weight per unit length $\bar{\psi}$ is given by the specific cost function

$$\bar{\psi} = \bar{k}\,|\,\bar{M}\,| + \bar{c}\,|\,\bar{V}\,|, \tag{5}$$

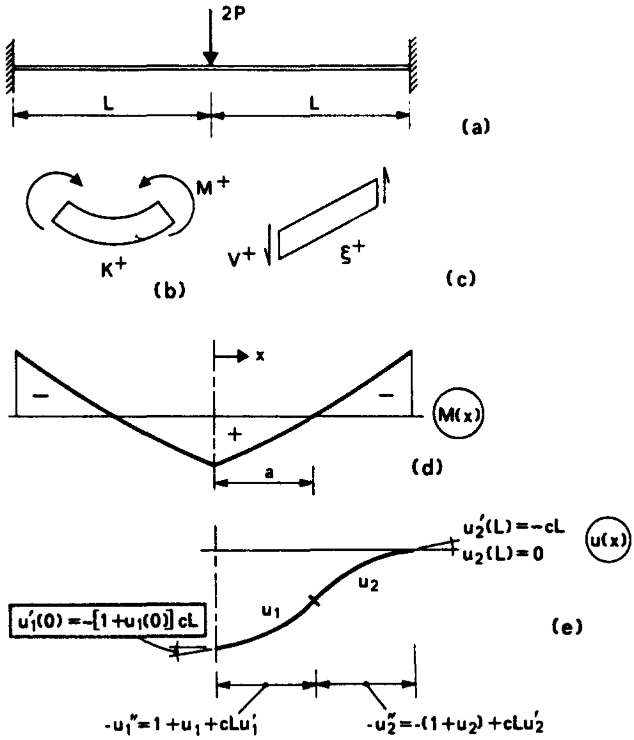

Fig. 2. Beam example: problem statement, sign conventions and optimality criteria.

where $\bar{k}$ and $\bar{c}$ are given constants, $\bar{M}$ is the beam bending moment and $\bar{V}$ is the shear force. Equation (5) is a realistic specific cost function for trusses (or I-beams) because the specific volume of the chords (flanges) is governed mostly by the bending moment $\bar{M}$ and that of the web members (web) by the shear force $\bar{V}$. We introduce the non-dimensional notation $M = \bar{k}^{1/2}\bar{M}/\bar{P}$, $x = \bar{k}^{1/2}\bar{x}$ (where $\bar{x}$ is the longitudinal coordinate), $V = \bar{V}/\bar{P}$, $L = \bar{L}\bar{k}^{1/2}$, $c = \bar{c}/\bar{k}\bar{L}$, $\psi = \bar{\psi}/\bar{k}^{1/2}\bar{P} = |M| + cL|V|$, $\Phi = \bar{\Phi}/\bar{P} = \int_D \psi \, dx$ (where Φ is the total beam weight).

It follows that the equilibrium equation becomes

$$M'' = -(|M| + cL|V|) = -(|M| + cL|M'|), \tag{6}$$

where primes denote differentiation with respect to x.

Derivation of optimality criteria from variational principles

Using first a variational formulation, the above problem can be expressed as

$$\min \Phi = \int_D [|M| + cL|V| + u(M'' + |M| + cL|V|)] \, dx, \tag{7}$$

where u is a Lagrangian multiplier.

Introducing the slack functions s_1 through s_4 and Lagrangian multipliers a_1 through a_4, the specific cost components $\psi_1 = |M|$ and $\psi_2 = cL|V| = cL|M'|$ can be incorporated in the integrand

$$\min \Phi = \int_D [(\psi_1 + \psi_2)(1 + u) + a_1(-\psi_1 + M + s_1) + a_2(-\psi_1 - M + s_2) + a_3(-\psi_2 + cLM' + s_3) + a_4(-\psi_2 - cLM' + s_4) + uM''] \, dx, \tag{8}$$

where

$$s_1 = 0 \quad (\text{for } M \geq 0), \qquad s_2 = 0 \quad (\text{for } M \leq 0),$$
$$s_3 = 0 \quad (\text{for } M' \geq 0), \qquad s_4 = 0 \quad (\text{for } M' \leq 0). \tag{9}$$

Then necessary conditions for minimality furnish ([3], pp. 18, 19, [24]) for variations of ψ_1, ψ_2, M and s_i:

$$a_1 + a_2 = 1 + u, \qquad a_3 + a_4 = 1 + u, \tag{10}$$

$$a_1 - a_2 - cL(a_3' - a_4') = -u'', \tag{11}$$

$$a_i = 0 \quad (\text{for } s_i \neq 0), \qquad a_i \geq 0 \quad (\text{for } s_i = 0). \tag{12}$$

It follows that

$$(\text{for } M \neq 0,\ V \neq 0) \qquad -u'' = \operatorname{sgn} M(1 + u) + \operatorname{sgn} VcLu', \tag{13}$$

$$(\text{for } M = 0, V \neq 0) \qquad -(1 + u) + \operatorname{sgn} VcLu' \leq -u'' \leq 1 + u + \operatorname{sgn} VcLu', \tag{14}$$

where $V = -M'$.

Moreover, transversality conditions together with eqns (8) and (10) ([13], pp. 21, 22) yield for built-in ends (with $\delta M \neq 0$):

$$-u' |_{\text{END}} = cL(1 + u) \operatorname{sgn} V, \tag{15}$$

and for both simple supports and built-in ends (with $\delta M' \neq 0$)

$$u |_{\text{END}} = 0. \tag{16}$$

It will be seen from the next section that the optimality conditions in eqns (13)–(16) reduce to those furnished by the modified Prager–Shield condition (3), if we interpret the quantity $u(x)$ as the Pragerian beam deflection.

Derivation of optimality criteria from the modified Prager–Shield condition

In using the Prager–Shield condition[5] and eqns (2) and (3) herein, the generalized strain components "corresponding" to the generalized stress vector $\mathbf{Q} = (M, V)$ are $\mathbf{q} = (\kappa, \xi)$ where κ is the curvature and ξ is the shear strain. The kinematically admissible Pragerian deflections u^k can then be split into two components $u = u_m + u_v$ such that

$$-u_m'' = \kappa \qquad \text{and} \qquad -u_v' = \xi. \tag{17}$$

The sign conventions for the generalized stresses and strains are summarized in Figs. 2(b) and 2(c). Considering the nondimensional specific cost function

$$\psi = |M| + cL|V|, \tag{18}$$

the modified Prager–Shield condition (3) furnishes the optimal strain–stress relations:

$$\kappa = \operatorname{sgn} M(1 + u) \quad (\text{for } M \neq 0), \qquad |\kappa| \leq 1 + u \quad (\text{for } M = 0),$$
$$\xi = cL \operatorname{sgn} V(1 + u) \quad (\text{for } V \neq 0), \qquad |\xi| \leq cL(1 + u) \quad (\text{for } V = 0). \tag{19}$$

Since $u'' = u_M'' + u_V''$, eqns (17) and (19) imply the optimality conditions in eqns (13) and (14). Moreover, kinematic admissibility implies eqns (15) and (16).

Optimization using optimality criteria

A feasible statically admissible moment diagram and the corresponding deflection diagram satisfying all optimality conditions are shown in Figs. 2(d) and 2(e). It can be checked readily that the conditions

$$(\text{for } a \leq x \leq L) \quad -u_2'' = -(1 + u_2) + cLu_2', \quad u_2(L) = 0, \quad u_2'(L) = -cL \tag{20}$$

imply

$$u_2 = (e^{\alpha(L-x)}/\beta)\{\beta \cosh[(L - x)\beta] + \alpha \sinh[(L - x)\beta]\} - 1, \tag{21}$$

with $\alpha = cL/2$, $\beta = (1 + \alpha^2)^{1/2}$. Moreover, the conditions

$$(\text{for } 0 \leq x \leq a) \quad -u_1'' = 1 + u_1 + cLu_1', \quad u_1'(0) = -[1 + u_1(0)]cL, \quad u_1(a) = u_2(a)$$

are fulfilled by the deflection field

$$u_1 = e^{\alpha(L-x)}[\lambda \cos(x\lambda) - \alpha \sin(x\lambda)]\Theta - 1,$$

with

$$\Theta = \{\beta \cosh[(L - a)\beta] + \alpha \sinh[(L - a)\beta]\}/\{\beta[\lambda \cos(a\lambda) - \alpha \sin(a\lambda)]\}, \qquad \lambda = (1 - \alpha^2)^{1/2}. \tag{22}$$

Then the slope continuity condition $u_1'(a) = u_2'(a)$ furnishes the following equation for determining the *optimal value* of a:

$$\Theta\{\alpha[\lambda \cos(\lambda a) - \alpha \sin(\lambda a)] + \lambda[\lambda \sin(\lambda a) + \alpha \cos(\lambda a)]\} = +\beta \sinh[(L - a)\beta] + 2\alpha \cosh[(L - a)\beta] + (\alpha^2/\beta) \sinh[(L - a)\beta]. \tag{23}$$

Check by primal formulation and differentiation

Considering the moment diagram in Fig. 2(d), equilibrium conditions require

$$(\text{for } 0 \leq x \leq a) \quad M_1'' = -M_1 + cLM_1' = -\psi, \quad M_1'(0) = -1, \quad M_1(a) = 0, \tag{24}$$

furnishing

$$M_1 = e^{\alpha x}[\tan(\lambda a) \cos(\lambda x) - \sin(\lambda x)]/[\lambda - \alpha \tan(\lambda a)]. \tag{25}$$

Moreover

$$(\text{for } a \leq x \leq L), \quad M_2'' = M_2 + cLM_2' = -\psi, \quad M_2(a) = 0, \quad M_2'(a) = M_1'(a), \tag{26}$$

implying after simplifications

$$M_2 = -(\lambda/\beta)\, e^{\alpha x}\{\sinh[\beta(x - a)]/[\lambda \cos(\lambda a) - \alpha \sin(\lambda a)]\}. \tag{27}$$

The total cost Φ can be calculated by integrating the specific cost along the beam length

$$\Phi = 2 \int_0^L (|M| + cLM')\, dx. \tag{28}$$

However, the total weight of the beam is also given by taking twice the shear force

(M') at the beam ends and subtracting the magnitude of the external load (nondimensionally, 2.0):

$$\Phi = -2M'(L) - 2. \tag{29}$$

The latter result can also be obtained by replacing the integrand in eqn (28) with $\psi = -M''$ and then integrating by parts.

After simplifications, both eqns (28) and (29) furnish

$$\Phi = 2\,e^{\alpha L}(\lambda/\beta)\{\beta\cosh[(a - L)\beta] + \alpha\sinh[(L - a)\beta]\}/[\lambda\cos(\lambda a) - \alpha\sin(\lambda a)] - 2. \tag{30}$$

The stationary condition $d\Phi/da = 0$ then confirms the result in eqn (23).

Check on the minimum total cost by dual formulation

In the considered problem, the complementary cost is zero ($\hat{\psi} = 0$) since $|\kappa| \leq (1 + u)$ and $|\xi| \leq cL(1 + u)$ [see Fig. 1(b)]. Hence the minimum total cost Φ_{min} is also given by [see eqn (4)]

$$\Phi_{min} = \int_D u^k p \, dx = (2.0)[u_1(0)]. \tag{31}$$

It can be checked readily that eqn (31) with eqn (22) furnish the same result as eqn (30).

The variation of the minimum total weight Φ_{min} as a function of the span L and shear cost factor c [see eqns (28), (29) or (31)] is given in Fig. 3, and the optimal values of a in Fig. 4.

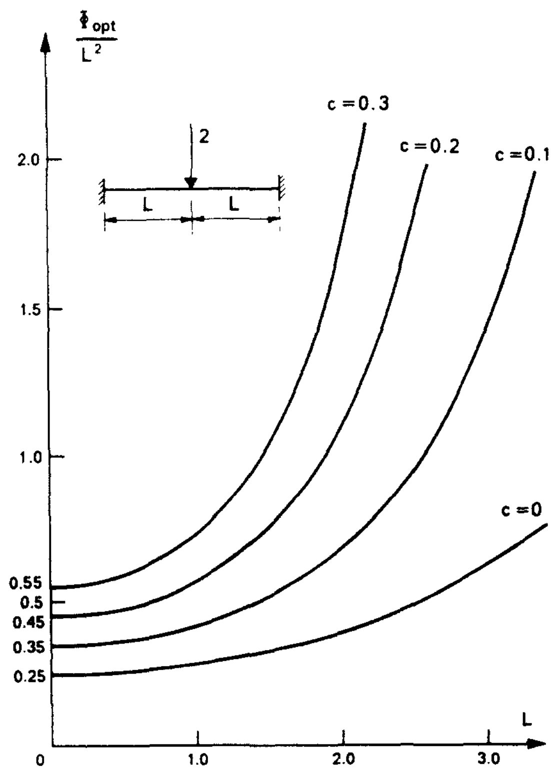

Fig. 3. Beam example: optimal cost values.

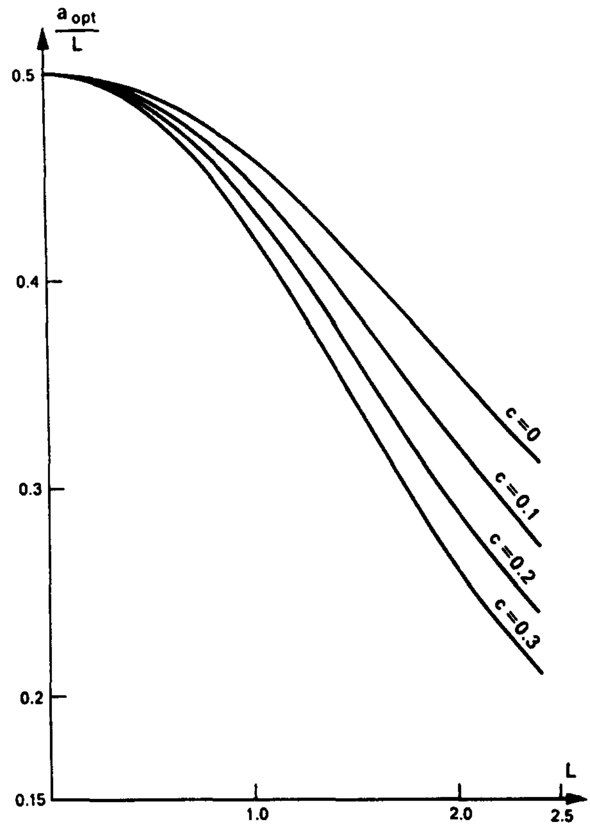

Fig. 4. Beam example: optimal distances of the zero moment point (a).

OPTIMALITY CRITERIA FOR LONG-SPAN TRUSS-GRIDS

Problem formulation

A vertical load system $p(x, y)$ defined on a horizontal plane domain D with coordinates (x, y) is to be transmitted to the boundary of D by means of a system of intersecting trusses. The distance between the two truss-chords is prescribed and constant throughout the system, and thus the specific cost function for all trusses is the one given in eqn (5).

The *structural universe* in this problem consists of trusses running in all horizontal directions across any arbitrary point (x, y) of the domain D. For trusses with nonzero moment and/or shear force, the optimality criteria are given in eqns (13) and (14). Moreover, for vanishing (nonoptimal) trusses, the optimality conditions become

$$(\text{for } M = 0, V = 0) \qquad -(1 + u) - cLu' \leq -u'' \leq 1 + u + cLu'. \qquad (32)$$

In the context of truss-grids, primes in eqns (13), (14) and (32) denote differentiation with respect to the coordinate along the truss axis (say w).

Using the above optimality criteria, the optimal solution can be determined for any boundary/loading condition. For *trusses without self-weight,* the optimality conditions in eqns (13), (14) and (32) [or in eqn (19)] become simpler [replace $(1 + u)$ by unity, and u' by zero]. For this simpler class of problems, solutions for various boundary shapes are already available [26]. Finally, if the specific cost is assumed to depend on the moment only ($\psi = \bar{k} \mid \overline{M} \mid$), then in eqns (13), (14) and (32) $c = 0$, and hence the corresponding terms vanish. For this simple truss grid (grillage) problem, a most comprehensive theory was developed by Rozvany[3, 19, 22–24] and Prager.

Considering now long-span truss grids with self-weight, applications of the general theory will be restricted in this paper to axially symmetric systems.

Axially symmetric truss-grids

We consider now an axially symmetric truss-grid (Fig. 5) in which radial and circumferential trusses resist the corresponding moments (M_r and M_θ in Fig. 5), and the shear force V is transmitted in the radial direction r. The radius of the edge of the truss grid is denoted by $\bar{R}$ and the following nondimensional notation is introduced:

$$r = \bar{r}\bar{k}^{1/2}, \quad R = \bar{R}\bar{k}^{1/2}, \quad M_i = \bar{k}\bar{M}_i/\bar{p} \quad (i = 1, 2),$$

$$V = \bar{V}\bar{k}^{1/2}/\bar{p}, \quad c = \bar{c}/\bar{k}\bar{R}, \quad \Phi = \bar{k}\bar{\Phi}/\pi\bar{p} = 2\int_0^R \psi r\, \mathrm{d}r,$$

$$\psi = \bar{\psi}/\bar{p} = |M| + cR|V|.$$

Then the static and kinematic conditions become

$$(rM_r)'' - M'_\theta = -r - \psi r, \tag{33}$$

$$\kappa_\theta = -u'/r, \quad \kappa_r = -u'', \tag{34}$$

where κ_θ and κ_r are the circumferential and radial curvatures, and u is the vertical deflection. The nondimensional curvatures represent $\kappa_i = \bar{\kappa}_i/\bar{k}$ ($i = \theta, r$) and the "associated" deflection $\bar{u}$ given by the Prager–Shield condition (2) is already nondimensional $u = \bar{u}$.

Optimality criteria for the above problem can be derived readily from conditions (13), (14) and (32) above. However, an additional direct derivation is also given herein. Incorporating the static (equilibrium) condition by means of a Lagrangian multiplier u, the extremum problem becomes

$$\min \Phi = 2\int_0^R \{(|M_\theta| + |M_r| + cRV) + u[(rM_r)'' - M'_\theta + r + |M_\theta| + |M_r| + cRV]\} r\, \mathrm{d}r. \tag{35}$$

Remembering that $rV = M_\theta - (rM_r)'$ and $(rM_r)'' = 2M'_r + rM''_r$, necessary conditions of minimality furnish [cf. eqns. (8)–(14) above]; (for $M_r \neq 0$)

$$\kappa_r = -u'' = (1 + u)\,\mathrm{sgn}\, M_r + cRu', \tag{36}$$

(for $M_r = 0$)

$$-(1 + u) + cRu' \leqslant -u'' \leqslant (1 + u) + cRu', \tag{37}$$

(for $M_\theta \neq 0$)

$$\kappa_\theta = -u'/r = (1 + u)(\mathrm{sgn}\, M_\theta + cR/r), \tag{38}$$

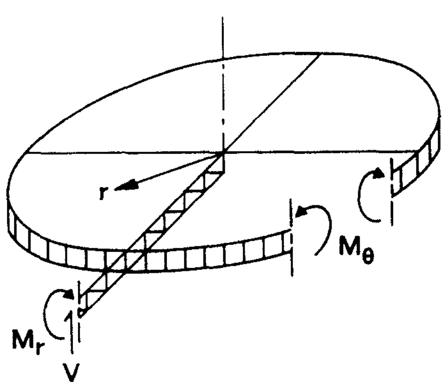

Fig. 5. Axially symmetric truss-grids: layout and stress resultants.

(for $M_\theta = 0$)

$$(1 + u)(-r + cR) \leq -u' \leq (1 + u)(r + cR), \tag{39}$$

where $u(r)$ is the Pragerian deflection.

OPTIMAL SOLUTIONS FOR CIRCULAR SIMPLY SUPPORTED TRUSS-GRIDS

Solution for intermediate span lengths: purely circumferential moment field

It will be shown in this section that for shorter spans ($R \leq R_{lim}$) the optimal solution consists of a purely circumferential moment field

$$M_r \equiv 0, \qquad M_\theta \geq 0. \tag{40}$$

Naturally, a system of one-way trusses only requires additional bracing to assure elastic stability. However, it has been established computationally and experimentally[27] that the structural weight of such bracing is relatively insignificant.

For the moment fields in eqn (40) with $u(R) = 0$, the condition (38) furnishes

$$-u'/r = (1 + u)(1 + cR/r), \qquad du/(1 + u) = -(r + cR)\,dr, \tag{41}$$

$$u = e^{-[r^2/2 + cRr - R^2(1/2 + c)]} - 1. \tag{42}$$

Since in this problem the complementary cost is zero ($\hat{\psi} \equiv 0$), eqn (4) furnishes

$$\Phi_{min} = 2\int_0^R ur\,dr = 2\{e^{R^2(1/2+c)} - 1 - cR\sqrt{\pi/2}\,e^{R^2(1+c)^2/2}\,[\mathrm{erf}\frac{R + cR}{\sqrt{2}} - \mathrm{erf}(cR/\sqrt{2})]\} - R^2 \tag{43}$$

where "erf" denotes the error function

$$\mathrm{erf}(r) = (2/\sqrt{\pi})\int_0^r e^{-t^2}\,dt.$$

However, for $M_r \equiv 0$, the optimality condition (37) must also be fulfilled for sufficiency of the optimal solution. Substituting eqn (42) into the relevant part of (37),

$$-(1 + u) + cRu' \leq u'', \tag{44}$$

we readily obtain

$$R \leq \sqrt{2/(1 + c)}, \qquad R_{lim} = \sqrt{2/(1 + c)}, \tag{45}$$

where R_{lim} is the limiting radius for the above type of optimal solutions.

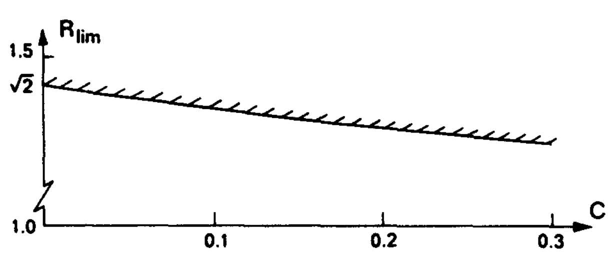

Fig. 6. Limiting radius (R_{lim}) of support for the optimal solution with purely circumferential moments as a function of the shear cost factor (c).

It will be shown in Part II that for $R > R_{lim}$, another type of solution becomes more economical than the one with $M_r \equiv 0$. The variation of the limiting radius R_{lim} in terms of the shear cost factor c is given in Fig. 6, and the corresponding Pragerian deflection fields are given in Fig. 7.

The moment field for the above solution is given by eqn (33) with $M_r \equiv 0$,

$$M_\theta' = r(1 + \psi) = r[1 + M_\theta(1 + cR/r)], \tag{46}$$

with the boundary condition $M_\theta(0) = 0$, eqn (46) then furnishes

$$\begin{aligned} M_\theta &= \left[-cR \int_0^r e^{-(r^2/2 + cRr)}\, dr + 1\right] \Big/ e^{-(r^2/2 + cRr)} - 1 \\ &= e^{(r^2/2 + cRr)} - \sqrt{\pi/2}\, cR\, e^{(r + cR)^2/2}\{\mathrm{erf}[(r + cR)/\sqrt{2}] - \mathrm{erf}(cR/\sqrt{2})\} - 1. \end{aligned} \tag{47}$$

The minimum total cost Φ_{min} is then given by

$$\Phi_{min} = 2\int_0^R r\psi\, dr = 2\int_0^R (rM_\theta + cRM_\theta)\, dr, \tag{48a}$$

or by

$$\Phi_{min} = 2RV(R) - R^2 = 2M_\theta(R) - R^2, \tag{48b}$$

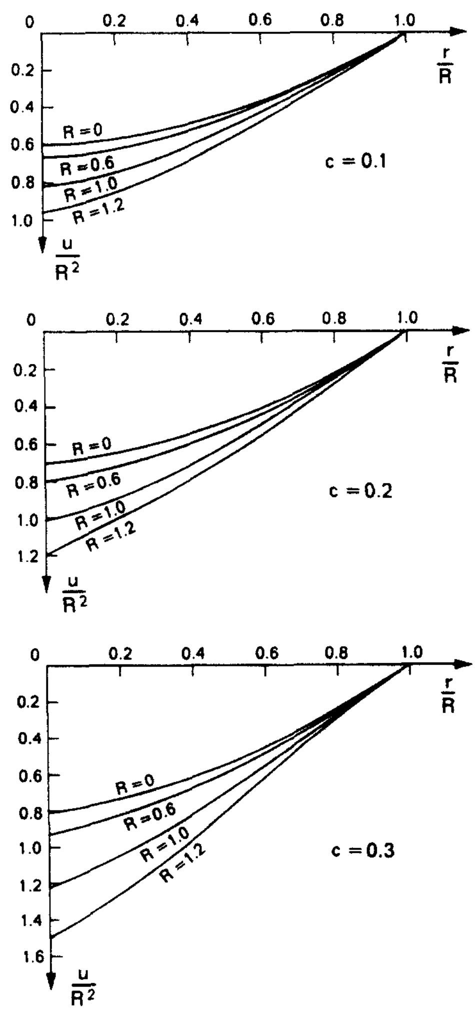

Fig. 7. Pragerian deflection fields for circular trusses with $R < R_{lim}$.

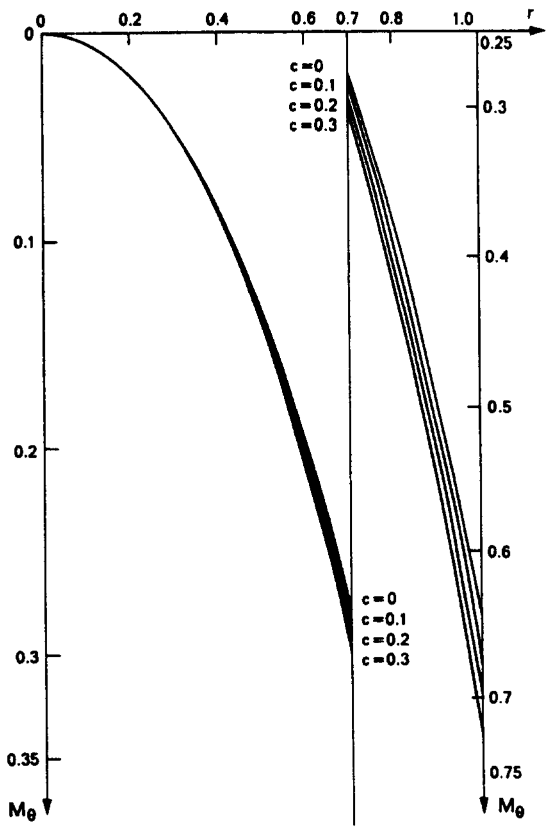

Fig. 8. Optimal moment fields for trusses with $R < R_{lim}$.

which confirm the result in eqn (43). The variation of $M_\theta(r)$ for various values of c is shown in Fig. 8.

Check for special cases

For $c = 0$, eqn (45) furnishes

$$R_{lim} = \sqrt{2}, \tag{49}$$

which confirms earlier results by Rozvany and Wang[2]. Moreover, for small values of the radius ($R \to 0$), a Taylor expansion of eqns (42) and (47), after neglecting infinitesimals of higher order, furnishes

$$u = (R^2 - r^2)/2, \qquad \kappa_\theta = -u'/r = \kappa_r = -u'' = 1, \tag{50}$$

$$M_\theta = r^2/2. \tag{51}$$

This result satisfies the optimality and equilibrium conditions (36), (38) and (33) without self-weight: (for $M_r > 0$),

$$\kappa_r = 1,$$

(for $M_\theta > 0$)

$$\kappa_\theta = 1,$$

(for $M_r \equiv 0$)

$$M_\theta' = r. \tag{52}$$

The solution in eqn (50) has been known for some time (e.g. [3], pp. 186, 187). It is interesting to note that for short spans without self-weight, (52) would admit any combination of positive radial and circumferential moments, whereas for longer spans with self-weight (with $R \leqslant R_{\lim}$) only $M_\theta \geqslant 0$, $M_r \equiv 0$ is optimal.

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APPENDIX

A. *Proof of identical result from primal and dual formulations: circular truss-grid.*

Primal formulation. Substitution of eqn (47) into eqn (48b) yields directly eqn (43).

Dual formulation. Substitution of eqn (42) into eqn (4) implies

$$\begin{aligned}\Phi &= 2\int_0^2 r(e^{-[r^2/2+cRr-R^2(1/2+c)]}-1)\,dr \\ &= 2\int_0^R (r+cR-cR)\,e^{-(r^2/2+cRr)}\,e^{R^2(1/2+c)}\,dr - R^2 \qquad \text{(A1)} \\ &= 2\,e^{R^2(1/2+c)}\int_0^R [(r+cR)\,e^{-(r^2/2+cRr)} - cR\,e^{-(r^2/2+cRr)}]\,dr.\end{aligned}$$

The first term in the above integrand can be integrated directly:

$$\int_0^R (r+cR)\,e^{-(r^2/2+cRr)}\,dr = -e^{-(R^2/2+cR^2)}+1, \qquad \text{(A2)}$$

and the second term is transformed into the following form:

$$e^{-(r^2/2+cRr)} = e^{(cR)^2/2}\,e^{-(r+cR)^2/2} \qquad \text{(A3)}$$

whose integral is an error function,

$$\operatorname{erf}(r) = (2/\sqrt{\pi})\int_0^r e^{-t^2}\,dt \qquad \text{(A4)}$$

with $t^2 = (r+cR)^2/2$.

The modified limits of integration then become

$$(r=0) \quad t_1 = cR/\sqrt{2}. \qquad (r=R) \quad t_2 = (R+cR)/\sqrt{2}. \qquad \text{(A5)}$$

Noting that

$$\int_{t_1}^{t_2} e^{-t^2\,dt} = \int_0^{t_2} e^{-t^2}\,dt - \int_0^{t_1} e^{-t^2}\,dt, \qquad \text{(A6)}$$

and $dr = \sqrt{2}\,dr$, integration of the last part of eqn (A3) furnishes

$$\int_0^R e^{(r+cR)^2/2}\,dr = \sqrt{\pi/2}\,\{\operatorname{erf}[(R+cR)/\sqrt{2}] - \operatorname{erf}(cR/\sqrt{2})\}. \qquad \text{(A7)}$$

Then eqn (A1) with eqns (A2), (A3) and (A7) implies eqn (43).

B. *Proof that differentiation of the primal cost and the associated displacement field furnish the same optimal* "a" *value*

The stationary condition $d\Phi/da = 0$ and eqn (30) imply

$$\frac{\lambda\cos(a\lambda) - \alpha\sin(a\lambda)}{(\alpha^2-1)\sin(a\lambda) - \alpha\lambda\cos(a\lambda)} = \frac{\beta\cosh[(a-L)\beta] + \alpha\sinh[(L-a)\beta]}{(1+\alpha^2)\sinh[(a-L)\beta] - \alpha\beta\cosh[(L-a)\beta]}, \qquad \text{(A8)}$$

which can also be obtained by rearranging eqn (23) in the form

$$\frac{\lambda[-\lambda\sin(a\lambda) - \alpha\cos(a\lambda)]}{\lambda\cos(a\lambda) - \alpha\sin(a\lambda)} - \alpha = \frac{\beta\{-\beta\sinh[(L-a)\beta] - \alpha\cosh[(L-a)\beta]\}}{\beta\cosh[(L-a)\beta] + \alpha\sinh[(L-a)\beta]} - \alpha. \qquad \text{(A9)}$$

Equation (A9) then implies eqn (A8).